

Bose and Einstein Meet Newton

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Abstract

We model the time evolution of a Bose-Einstein condensate, subject to a special periodically excited optical lattice, by a unitary quantum operator U on a Hilbert space H . If a certain parameter $\alpha = p/q$, where p and q are coprime positive integers, then $H = L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C}^q)$ and U is represented by a $q \times q$ matrix-valued function M on \mathbb{R}/\mathbb{Z} that acts pointwise on functions in H . The dynamics of the quantum system is described by the eigenvalues of M . Numerical computations show that the characteristic polynomial $\det(zI - M(t)) = \prod_{j=1}^q (z - \lambda_j(t))$ where each λ_j is a real analytic function that has period $1/q$. We discuss this phenomena using Newton's Theorem, published in *Geometria analytica* in 1660, and modern concepts from analytic geometry.

Keywords: bose-einstein condensate, unitary quantum operator, characteristic polynomial, newton's theorem, resolution of singularities, étale homotopy.

1 Introduction and Preliminary Results

This paper discusses polynomials, which are used to model the dynamics of certain quantum systems involving Bose-Einstein condensates controlled by optical lattices, and that arise as follows.

Let $\alpha = p/q$ where $p = 2$ and $q = 5$, and construct the family of matrix valued functions $M_\kappa(t) = D_\kappa(t) G^{-1} D_\kappa(t) G$, $\kappa > 0$ where

$$G = \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 \\ g_4 & g_0 & g_1 & g_2 & g_3 \\ g_3 & g_4 & g_0 & g_1 & g_2 \\ g_2 & g_3 & g_4 & g_0 & g_1 \\ g_1 & g_2 & g_3 & g_4 & g_0 \end{bmatrix}, \quad D_\kappa(t) = \begin{bmatrix} c_0(t) & 0 & 0 & 0 & 0 \\ 0 & c_1(t) & 0 & 0 & 0 \\ 0 & 0 & c_2(t) & 0 & 0 \\ 0 & 0 & 0 & c_3(t) & 0 \\ 0 & 0 & 0 & 0 & c_4(t) \end{bmatrix},$$

$$g_j = \frac{1}{q} \sum_{k=0}^{q-1} e^{-i2\pi k^2 \alpha} e^{i2\pi j k \alpha} \quad \text{and} \quad c_j(t) = e^{-i2\pi \kappa \cos 2\pi(t-j\alpha)} \quad \text{for } j = 0, 1, 2, 3, 4.$$

As explained in Appendix A, the matrix valued function M_κ represents a Floquet operator that describes the time evolution of a quantum system called an *on resonance double kicked rotor*. The characteristic polynomial $C(t, z) = \det(zI - M_\kappa(t))$ can be regarded as a polynomial of degree 5 in the z variable whose coefficients are functions of the t variable that are analytic and have period 1. A closer inspection of the structure of M_κ reveals that each coefficient of $C(t, z)$ also has period $\alpha = p/q$ and therefore has period $1/q$. Therefore there exists a polynomial $P(t, z)$ of z whose coefficients are functions of t that have period 1 such that $P(qt, z) = C(t, z)$. The dynamics of the system is described by the spectrum of the Floquet operator which equals $\cup_{t=0}^1 \text{roots} P(t, z)$. This implies that the spectrum consists of a union of at most $q = 5$ disjoint closed intervals or *bands*. Numerical investigations show that for sufficiently small values of κ there are q bands for odd q and $q - 1$ bands for even q . This can be shown to imply that each root of $P(t, z)$ is a continuous period 1 function of t . As κ increases the crossing of the graphs of the roots increases and for sufficiently large κ there is only 1 band. However, if we fix the value of κ and construct matrices corresponding to rational $\alpha = p/q$ as α approaches an irrational

number, and therefore $q \rightarrow \infty$, we find that more bands appear. Based on extensive numerical experiments we conjectured in a previous paper [31] that the spectrum approaches a Cantor set. Strong support for our Cantor conjecture would be provided by

Conjecture 1 *If α is rational and $\kappa > 0$ and P is constructed as above for the Floquet operator corresponding to α and κ , then each root of $P(t, z)$ is an analytic function of t that has period 1.*

This conjecture is supported by extensive numerical computation. The results in this paper imply by a homotopy argument that if the roots of $P(t, z)$ are locally analytic functions then the conjecture holds. They also imply that if the multiplicity of the roots is at most 2 then the conjecture holds. A proof of this conjecture would support our Cantor conjecture since if each root of $P(t, z)$ is an analytic function of period 1 then each root of $C(t, z)$ is analytic and has period $1/q$ so this gives a small upper bound on the range of each root. Since this is a pure mathematics paper we summarize the physical background in Sections 5 and 6 where references for further study are suggested.

2 Preliminary Results

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the integer, rational, real, and complex numbers, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the real circle group. For any algebra A , $A[z]$ denotes the algebra of polynomials with coefficients in A . A polynomial is called *monic* if the coefficient of $z^{\deg(P)}$ equals 1, *simple* if there does not exist $Q \in A[z]$ with $\deg(Q) \geq 1$ such that Q^2 divides P , *irreducible* if P does not admit a factorization $P = QR$ where $\deg(Q) \geq 1$ and $\deg(R) \geq 1$, and *completely reducible* (CR) if it admits a factorization $P(z) = \prod_{j=1}^{\deg P} (z - a_j)$ with $a_j \in A$. $K = C^\omega(\mathbb{T})$ denotes the algebra of analytic functions on \mathbb{T} and $K_0 = C_0^\omega(\mathbb{T})$ denotes the algebra of *germs of analytic functions* at 0 (power series in t that converge absolutely for t sufficiently small). Clearly $K \subset K_0$ and $K \neq K_0$. For $P \in K[z]$, $P_0 \in K_0[z]$ denotes the element obtained by regarding the coefficients of P to be in K_0 . \mathbb{T} acts as a group of algebra automorphisms of $K[z]$ by rotation $(R_s P)(t, z) = P(t + s, z)$. For $P \in K[z]$ and $s \in \mathbb{T}$ we define $P_s = (R_s P)_0 \in K_0[z]$. We call $P \in K[z]$ *locally completely reducible* (LCR) if for every $s \in \mathbb{T}$, $(R_s P)_0 \in K_0[z]$ is CR.

Example 1 *If $P(t, z) = z^2 - \sin^2 2\pi t$ then $P(0, z) \in \mathbb{C}[z]$ is not simple. However $P \in K[z]$ and $P_0 \in K_0[z]$ are simple.*

Question 1 *When is a monic polynomial in $A[z]$ simple for $A = \mathbb{C}$, $A = K$, $A = K_0$?*

Example 2 *$P(t, z) = z^2 - \cos 2\pi t \in K[z]$ is not LCR since $P_{\pm 1/2} \in K_0[z]$ are not CR.*

Question 2 *When is a monic polynomial $P \in K[z]$ LCR?*

Example 3 *If $P(t, z) = z^2 - e^{2\pi i t} \in K[z]$ then P is LCR but not CR.*

Question 3 *When is a LCR monic polynomial $P \in K[z]$ CR?*

If A is one of the algebras \mathbb{C} , K , or K_0 , then $A[z]$ is a *Euclidean domain*, therefore $P(z) = p_m z^m + p_{m-1} z^{m-1} + \cdots + p_1 z + p_0$ and $Q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1 z + q_0$ in $A[z]$ have a *greatest common divisor* $\text{GCD}(P, Q)$. Furthermore, the *Euclidean Algorithm*, codified about 300 BCE by Euclid in Books VII and X in his Elements [23], but likely known to Eudoxus of Cnidus about 375 BCE [7], for computing the greatest common divisor of two positive integers, can be also used to compute $P_1, Q_1 \in A[z]$ such that $\text{GCD}(P, Q) = P_1 P + Q_1 Q$ (perhaps the first Bezout identity [8]). This gives an algorithmic solution to Question 1. We now give an explicit solution. The *Sylvester Resultant* $R(P, Q)$

is the determinant of the following $(m+n) \times (m+n)$ matrix

$$\begin{bmatrix} p_m & p_{m-1} & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & p_1 & p_0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_m & p_{m-1} & \cdot & \cdots & \cdots & \cdot & \cdot & \cdot & p_1 & p_0 \\ q_n & q_{n-1} & \cdot & q_1 & q_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & q_n & q_{n-1} & \cdot & q_1 & q_0 \end{bmatrix}.$$

In 1840 Sylvester proved [38] that if $A = \mathbb{C}$ and $p_m \neq 0$ and $q_n \neq 0$ then $R(P, Q) = 0$ if and only if P and Q have a common root, or equivalently, $\deg(\text{GCD}(P, Q)) \geq 1$. It follows directly that this result also holds for $A = K$ and $A = K_0$. The *derivative* of $P \in A[z]$ is the polynomial $P'(z) = mp_m z^{m-1} + \cdots + 2p_2 z + p_1 \in A[z]$, and the *discriminant* of a monic $P \in A[z]$ with $\deg(P) \geq 2$ is $D(P) = -R(P, P')$. For example

$$D(z^2 + p_1 z + p_0) = -\det \begin{bmatrix} 1 & p_1 & p_0 \\ 2 & p_1 & 0 \\ 0 & 2 & p_1 \end{bmatrix} = p_1^2 - 4p_0. \quad (1)$$

For monic $P \in \mathbb{C}[z]$, $D(P) = -\prod_{i \neq j} (\lambda_i - \lambda_j)$, λ_i are roots of P , ([30], Proposition 10.5).

Lemma 1 *A monic polynomial $P \in \mathbb{C}[z]$ is not simple iff $D(P) = 0$. A monic polynomial $Q \in K_0[z]$ is not simple iff $D(Q) = 0$ (this means that $D(Q)(t) = 0$ for t sufficiently small). A monic polynomial $P \in K[z]$ is not simple iff $P_0 \in K_0[z]$ is not simple.*

Proof The first and second assertions follow since P is not simple iff P and P' have a common factor with degree ≥ 1 . The third assertion follows from the facts that $D(P(t, z))$ is an analytic function of t and $D(P(t, z)) = D(P_0)(t)$ so if it vanishes for t sufficiently small then it vanishes for all $t \in \mathbb{T}$.

For the polynomial $P = z^2 - \sin^2 2\pi t$ in Example 1, since $D(P) = 4 \sin^2 2\pi t$ vanishes at $t = 0$ but does not vanish in a neighborhood of 0, $P(0, z)$ is simple but P and P_0 are not simple. Lemma 1 gives a complete answer to Question 1 and reduces Questions 2 and 3 to questions about simple monic polynomials. Furthermore, the following result provides a partial answer to Question 2.

Lemma 2 *If $Q \in K_0[z]$ is monic and $Q(0, z) \in \mathbb{C}[z]$ is simple then Q is CR.*

Proof Since each root μ of $Q(0, z)$ has multiplicity 1,

$$Q'(0, z) = \frac{\partial Q}{\partial z}(0, \mu) \neq 0, \quad (2)$$

and hence the Implicit Function Theorem (for Holomorphic Functions of Several Complex Variables) ([22], Chapter 1. Theorem 4) implies that there exists $\eta_\mu \in K_0$ such that $\eta_\mu(0) = \mu$ and $Q(t, \eta_\mu(t)) = 0$ for t sufficiently small. Then $Q(t, z) = \prod_\mu (z - \eta_\mu(t))$ where the product is over the roots of $Q(0, z)$ so Q is CR.

Corollary 1 *If $P(t, z) = z^2 + p_1(t)z + p_0(t) \in K_0[z]$ has real valued coefficients and its roots have modulus 1 then P is CR.*

Proof Since the roots are complex conjugates of each other $p_0 = 1$ and $|p_1(t)| \leq 2$. Therefore the roots have the form

$$\lambda_{\pm}(t) = -\frac{1}{2}p_1 \pm \frac{1}{2}i\sqrt{4 - p_1^2(t)}.$$

If $p_1^2(0) < 4$ then the roots are distinct so Lemma 2 implies that $\lambda_{\pm} \in K_0$. Otherwise $p_1^2(t)$ has a maximum value of 4 at $t = 0$ and hence either $p_1^2(t) = 4$ for all t sufficiently small or there exists $c > 0$ and positive integer k such that $p_1^2(t) = 4 - ct^{2k} + \text{higher order terms}$. Therefore $\lambda_{\pm} \in K_0$. In either case $P(z) = (z - \lambda_+)(z - \lambda_-)$ is CR.

Definition 1 We call a monic polynomial $Q \in \mathbb{C}[z]$ *primary* if $Q(z) = (z - \mu)^m$ for some $\mu \in \mathbb{C}$. A monic polynomial $Q \in K_0[z]$ is called *point primary (PP)* if $Q(0, z)$ is primary. A factorization of $Q = Q_1 \cdots Q_n$, $Q_j \in K_0[z]$ is called a *point primary factorization (PPF)* if each factor Q_j is PP and $Q_1(0, z), \dots, Q_n(0, z) \in \mathbb{C}[z]$ have distinct roots.

$Q \in \mathbb{C}[z]$ is primary iff the *principal ideal* $(Q) = \{QP : P \in \mathbb{C}[z]\}$ is a *primary ideal*. The factorization of Q into primary factors corresponds to the *primary decomposition* of (Q) , which is a primary topic in commutative algebra ([4], Chapter 4), ([17], Chapter 3), ([29], Chapter VI. Section 2), ([30], Chapter VI, Section 5). However, PPF does not correspond to primary decomposition in $K_0[z]$. For any polynomial $Q \in \mathbb{C}[z]$, let $\Lambda(Q)$ denote the set of (distinct) roots of Q and for $\mu \in \Lambda(Q)$, let $m(\mu)$ denote the multiplicity of μ .

Theorem 1 (Hensel) Every monic $Q \in K_0[z]$ admits a PPF.

Proof Consider the primary factorization

$$Q(0, z) = \prod_{\mu \in \Lambda(Q(0, z))} (z - \mu)^{m(\mu)}. \quad (3)$$

Since the factors $(z - \mu)^{m(\mu)}$, $\mu \in \Lambda(Q(0, z))$ are pairwise relatively prime, Hensel's lemma [24], proved in 1908, implies that for every $\mu \in \Lambda(Q(0, z))$ there exists $Q_{\mu} \in K_0[z]$ such that $Q_{\mu}(0, z) = (z - \mu)^{m(\mu)}$ and $Q = \prod_{\mu \in \Lambda(Q(0, z))} Q_{\mu}$. This concludes the proof. Abhyankar ([1], 90–92) gives an algebraic proof of Hensel's lemma for polynomials with coefficients in the algebra of formal power series in an arbitrary field and gives an exercise ([1], p. 92) that implies Hensel's lemma holds in $K_0[z]$. We give an analytic proof using *Cauchy's Residue Formula*, which he presented to the Academy of Sciences of Turin in 1831. Let

$$r = \frac{1}{3} \min\{|\mu - \xi| : \mu, \xi \in \Lambda(Q(0, z)), \mu \neq \xi, \}. \quad (4)$$

For each $\mu \in \Lambda(Q(0, z))$ let $\Omega_{\mu} = \{z \in \mathbb{C} : |z - \mu| < r\}$, construct the following circular contour oriented counterclockwise $\Gamma_{\mu} = \{\mu + r e^{i\theta} : \theta \in [0, 2\pi)\}$, choose $\delta_{\mu} > 0$ so that $|Q(t, z)| > 0$ whenever $t \in (-\delta_{\mu}, \delta_{\mu})$ and $z \in \Gamma_{\mu}$. Then construct

$$I_{\mu}(t, z) = \frac{1}{2\pi i} \int_{w \in \Gamma_{\mu}} \frac{Q'(t, w)}{Q(t, w)} \frac{1}{z - w} dw, \quad t \in (s - \delta, s + \delta), \quad z \notin \Omega_{\mu} \cup \Gamma_{\mu}, \quad (5)$$

and

$$Q_{\mu}(t, z) = \prod_{\lambda \in \Lambda(Q(0, z)) \cap \Omega_{\mu}} (z - \lambda)^{m(\lambda)}, \quad t \in (-\delta_{\mu}, \delta_{\mu}), \quad z \in \text{complement of } \Omega_{\mu} \cup \Gamma_{\mu}. \quad (6)$$

Cauchy's residue formula implies that

$$I_{\mu}(t, z) = \frac{Q'_{\mu}(z)}{Q_{\mu}(z)} = \frac{d}{dz} \log Q_{\mu}(t, z) \quad (7)$$

and $Q(t, z) = \prod Q_{\mu}(t, z)$ whenever $t \in \cap_{\mu \in \Lambda(Q(0, z))} (-\delta_{\mu}, \delta_{\mu})$ and z is in the complement of $\cup_{\mu \in \Lambda(Q(0, z))} \Omega_{\mu} \cup \Gamma_{\mu}$. Since each $I_{\mu}(t, z)$ is an analytic function of t it follows that each $Q_{\mu} \in K_0[z]$. This concludes the proof.

Corollary 2 A monic $Q \in K_0[z]$ is CR iff each of its primary factors is CR.

Lemma 1 shows that if $Q \in K_0[z]$ is not simple then we may use the Euclidean algorithm to factorize it into simple factors. This fact in combination with Theorem 1 can be used to reduce Question 2 to

Question 4 When is a monic simple PP polynomial $Q \in K_0[z]$ CR?

Lemma 1, in combination with the obvious fact that if $P \in K[z]$ is CR then P is LCR, can be used to reduce Question 3 to

Question 5 When is a LCR monic simple polynomial $P \in K[z]$ CR?

3 Meet Newton

The objective of this section is to give (at least) a partial answer to Question 4.

Theorem 2 (Newton) If $Q \in K_0[z]$ is a monic polynomial of degree $n > 0$ then there exists a positive integer m that divides $n!$ and $\eta_1, \dots, \eta_m \in K_0$ and $\delta > 0$ such that

$$Q(t^m, z) = \prod_{j=1}^m (z - \eta_j(t)), \quad t \in (-\delta, \delta). \quad (8)$$

If Q is irreducible then $m = n$ and the roots can be labeled so that $\eta_j(t) = \eta_m(e^{2\pi i j/m} t)$.

Proof These two statements are the Supplement 1 and Supplement 2 cases of Newton's Theorem that Abhyankar proves in ([1], 89-98) using Hensel's lemma and Newton's generalized binomial theorem. He also remarks: "Newton proved this theorem about 1660 [35]. It was revived by Puiseux in 1850 [36]. The relevant history can be found in G. Chrystal's *Textbook of algebra*, vol. 2, 396 [16]."

Example 4 If $Q = z^2 + q_1 z + q_0 \in K_0[z]$ then there exists $k \geq 0$, $c \neq 0$, and $g \in K_0$ such that $D(Q)(t) = t^k(c + tg(t))$ and $Q(t^2, z) = (z - \eta_1(t))(z - \eta_2(t))$ where

$$\eta_j(t) = -a_1(t^2)/2 + (-1)^j t^k (c + t^2 g(t^2))^{1/2}, \quad j = 1, 2.$$

Therefore Q is CR iff k is even and is irreducible iff k is odd. Hence if $P(z) = z^2 - \cos 2\pi t$ is the polynomial in Example 2, then $P_{\pm 1/2} \in K_0[z]$ is irreducible.

Equation 8 gives an analytic parameterization for the analytic set $\{(t, z) : Q(t, z) = 0\}$. This is an example of a *resolution of singularities* that has been a central story in algebraic geometry leading to Hironaka's seminal 1964 paper [25]. Abhyankar [3] gives a fascinating account of this story. Theorem 2 implies that a monic polynomial $Q \in K_0[z]$ is CR iff the Taylor series of each η_j in Equation 8 only has terms ct^k where m divides k . Unfortunately, this fact is not very useful because it does not provide an algorithm to compute the η_j . We need more help from Newton.

For any $f \in K_0$ with $f \neq 0$ let $\text{ord}(f) \in \{0, 1, 2, \dots\}$ denote the smallest integer ℓ such that $f^{(\ell)}(0) \neq 0$. If $\eta(t)$ is a formal power series in nonnegative fractional powers of t and $\eta \neq 0$, $\text{ord}(\eta)$ denotes the smallest power of t in the power series expansion. Such η do not usually belong to K_0 . However, every element K_0 can be identified with the formal power series defined by its Taylor expansion. Examples: $\text{ord}(\cos 2\pi t) = 0$, $\text{ord}(\sin 2\pi t) = 1$, $\text{ord}(t^{1/2} + t - t^{4/3}) = 1/2$.

Definition 2 For $Q(z) = q_m z^m + q_{m-1} z^{m-1} + \dots + q_1 z + q_0 \in K_0[z]$ with $q_m = 1$ and $q_0 \neq 0$ let $\mathfrak{P}(Q) = \{(j, \text{ord}(q_j)) : q_j \neq 0, j = 0, \dots, m\}$. The Newton Polygon $\mathfrak{N}(Q)$ of Q is the convex hull in \mathbb{R}^2 of $\mathfrak{P}(Q)$. Let $\mathfrak{E}(Q)$ denote the extreme points of $\mathfrak{N}(Q)$. Then $\mathfrak{E}(Q) \subseteq \mathfrak{P}(Q)$, $(0, \text{ord}(q_0)) \in \mathfrak{E}(Q)$, and $(m, \text{ord}(q_m)) = (m, 0) \in \mathfrak{E}(Q)$. Starting with $(0, \text{ord}(q_0))$ we traverse the points in $\mathfrak{E}(Q)$ in a counterclockwise direction until we reach $(m, 0)$ to obtain $k+1$ points $(x_0, y_0) = (0, \text{ord}(q_0))$, $(x_1, y_1), \dots, (x_k, y_k) = (m, 0)$ where $k \leq m$. This gives positive integers $m_j = x_j - x_{j-1}$, $j = 1, \dots, k$ which satisfy $m_1 + \dots + m_k = m$ and slopes $s_j = (y_j - y_{j-1})/m_j$, $j = 1, \dots, k$ which satisfy $s_1 < s_2 < \dots < s_k \leq 0$.

Theorem 3 (Newton) For $j = 1, \dots, k$, $Q(t, z)$ has formal power series roots (with possible multiplicity > 1) $\eta_\ell(t)$, $\ell = 1, \dots, m_j$ that satisfy $\text{ord}(\eta_\ell) = -s_j$.

Proof This well known result was derived by Newton in his *Methodus Fluxionum et Serierum infinitarum* between 1664 et 1671 and translated into English by John Colson in 1736. See Chrystal's historical note in ([16], Part II, p. 396) and Harold Edward's essay [20]. The second assertion in the following result is analogous to the PPF in Theorem 1.

Corollary 3 *If $Q \in K_0[z]$ is irreducible then in Theorem 3, $k = 1$, $m_1 = m$, and there exists $\eta \in K_0$ with $\text{ord}(\eta) = y_1 - y_0 = \text{ord}(q_0)$ such that $\eta_\ell(t^m) = \eta(e^{2\pi i \ell/m} t)$. Furthermore, $\prod_{\ell=1}^{m_j} (z - \eta_\ell(t)) \in K_0[z]$.*

Proof The first assertion follows directly from Theorems 2 and 3. The second assertion follows from the fact that if Q is factored into irreducible factors in $K_0[z]$, then in Theorem 3 for $j = 1, \dots, k$, m_j is the sum over the irreducible factors P of Q of the number of formal power series roots ξ of P such that $\text{ord}(\xi) = s_j$.

The converse of the first assertion in Corollary 3 was disproved by Abhyankar ([1], 185–186) by constructing a reducible polynomial whose Newton polygon is a straight line. Anhyankar ([1], p. 185) gives necessary and sufficient criteria for irreducibility in $K_0[z]$ and give a comprehensive treatment of the question of irreducibility in [2]. The discussions in this section give (at least) a partial answer to Question 4 and hence to Question 2.

4 From Jets to Braids

The objective of this section is to completely answer Question 5 and thus Question 2.

Definition 3 *For every integer $k \geq 0$ we define the k -jet function $J_k : K_0 \rightarrow \mathbb{C}^{k+1}$ by*

$$J_k(f) = (f(0), f'(0), f^{(2)}(0), \dots, f^{(k)}(0)), \quad f \in K_0. \quad (9)$$

Lemma 3 *If $P \in K_0[z]$ is monic, simple, CR, and $\deg(P) = n$ then there exists an integer $k \geq 0$ such that $J_k(\lambda_1), \dots, J_k(\lambda_n)$ are distinct where $\lambda_1, \dots, \lambda_n \in K_0$ are the roots of P .*

Proof Assume to the contrary that for every integer $k \geq 0$, $J_k(\lambda_1), \dots, J_k(\lambda_n)$ are not distinct. Then there exists $1 \leq i < j \leq n$ such that λ_i and λ_j and all of their derivatives have the same value at 0. Since the roots are analytic $\lambda_i = \lambda_j$ so $(z - \lambda_i)^2$ divides P and hence P is not simple contrary to our assumption. This contradiction concludes the proof.

Example 5 $P(t, z) = z^2 - e^{2\pi i t}(1 + e^{2\pi i t})z + e^{6\pi i t} \in K_0[z]$ is simple and CR but $P(0, z) = (z - 1)^2 \in \mathbb{C}[z]$ is not simple. However $J_1(e^{2\pi i t}) = (1, 2\pi i) \neq J_1(e^{4\pi i t}) = (1, 4\pi i)$.

Corollary 4 *If $P \in K[z]$ is monic, simple, and LCR with degree $n \geq 2$, then there exists an integer $k \geq 0$ such that for every $s \in \mathbb{T}$, $J_k(\lambda_1), \dots, J_k(\lambda_n)$ are distinct where $\lambda_1, \dots, \lambda_n$ are the roots of $P_s \in K_0[z]$.*

Proof Since the roots of $P_s \in K_0[z]$ are analytic functions of s their k -jets are continuous. The result then follows since \mathbb{T} is compact.

Definition 4 *For any integer $n \geq 1$ and metric space space X let $C_n X$ denote the metric space consisting of subsets of X having n elements with the Hausdorff metric and let $x \in C_n X$. The fundamental group $\pi_1(C_n X, x)$ of C_n with base point x is called the braid group on X with base point x and denoted by $B_n(X, x)$. Let S_x denote the permutation group on x . We construct a canonical homomorphism $\Phi : B_n(X, x) \rightarrow S_x$ as follows: let $\phi : [0, 1] \rightarrow \mathbb{T}$ be the canonical group homomorphism, let $g \in B_n(X, x)$, and let $h : \mathbb{T} \rightarrow C_n X$ such that g is the homotopy class of h and $h(0) = h(1) = x$. Then the composition $f = h \circ \phi : [0, 1] \rightarrow C_n X$ satisfies $f(0) = f(1) = x$ and therefore induces a continuous map $F : [0, 1] \times X \rightarrow X$ such that $f(t) = \{F(t, u) : u \in x\}$. We observe that $F(1, \cdot)$ belongs to S_x and does not depend on the representative h for the homotopy class g . Then we define $\Phi(g) = F(1, \cdot)$. The kernel of Φ , called the pure braid group, is denoted by $P_n(X, x)$.*

We note the well known fact that $C_n \mathbb{C}$ is the configuration space that parameterizes the set of monic polynomials with n distinct roots ([37], page 15) and that braid groups arise in both classical mechanics [13] and quantum physics [15], [26]. The following result gives a complete answer to Question 5.

Theorem 4 Assume that $P \in K[z]$ is LCR, monic, simple, and $\deg(P) = n \geq 2$. Choose an integer $k \geq 0$ as in Corollary 4, define $h : \mathbb{T} \rightarrow C_n \mathbb{C}^{k+1}$ by

$$h(s) = \{ J_k(\lambda) : \lambda \in K_0, P_s(\lambda) = 0 \},$$

let $x = h(0)$, and define $B(P) \in B_n(C_n \mathbb{C}^{k+1}, x)$ to be the homotopy class of h . Then P is CR iff $B(P) \in P_n(C_n \mathbb{C}^{k+1}, x)$.

Proof Lemma 3 and Corollary 4 imply that $B(P)$ is an element in the braid group. The last assertion follows since the roots of P_s , $s \in \mathbb{T}$ in K_0 can be *glued together* to form roots of P in K precisely when $B(P)$ is an element in the pure braid subgroup.

Corollary 5 If $h : \mathbb{T} \rightarrow C_n \mathbb{C}^{k+1}$ is as in Theorem 4 then there exists a subset $S \subset \mathbb{C}^{k+1}$ of Lebesgue measure zero such that the Hermitian product $v * h : \mathbb{T} \rightarrow C_n \mathbb{C}$. Therefore $v * h$ gives an element in the braid group $B_n(\mathbb{C}, v * x)$.

Proof The first assertion follows from Sard's theorem and the second assertion is obvious.

Since elements of the braid group are invariant under homotopies of LCR polynomials, it follows that Conjecture 1 holds for any continuous family P_κ of LCR polynomials if it holds for any value of κ . We note that other homotopy invariants, such as Chern classes, have proved useful in the study of both the Integer Quantum Hall Effect [10] and the Fractional Quantum Hall Effect [33].

5 Origin of Questions in Quantum Physics

The questions in this paper arose from a study of the spectrums of Floquet operators that describe the dynamics of certain quantum systems. These operators include five families parameterized by parameters $\alpha \in (0, 1)$, $\lambda > 0$, $\kappa > 0$, $\theta \in (0, 1)$.

1. Harper (H) operators or Almost Mathieu operators. These self-adjoint operators can be represented on the Hilbert space $L^2(\mathbb{T})$ with respect to the standard orthonormal basis $\{\xi_n(t) = e^{2\pi i n t} : n \in \mathbb{Z}\}$ by $H(\alpha, \lambda, \theta)\xi_n = \xi_{n-1} + \xi_{n+1} + 2\lambda \cos(2\pi(n\alpha + \theta))$.
2. Unitary Harper (UH) operators $\exp[-i\kappa H(\alpha, \lambda, \theta)]$.
3. Kicked Harper (KH) operators $\exp[-i2\kappa \cos(2\pi t)] \exp[-i2\kappa \lambda \cos(-i\alpha \frac{d}{dt} + 2\pi\theta)]$.
4. Single Kicked Rotator (SKR) operators $\exp[-i2\kappa \cos(2\pi t)] \exp\left[\frac{i\alpha}{4\pi} \frac{d^2}{dt^2}\right]$.
5. On Resonance Double Kicked Rotator (ORDKR) operators $\exp[-i2\kappa \cos(2\pi t)] \exp\left[-\frac{i\alpha}{4\pi} \frac{d^2}{dt^2}\right] \exp[-i2\kappa \cos(2\pi t)] \exp\left[\frac{i\alpha}{4\pi} \frac{d^2}{dt^2}\right]$.
6. For rational $\alpha = p/q$ where p and q are coprime integers ≥ 2 , Mother operators constructed from each of these families of operators by forming their directed integral over $\theta \in [0, 1/q]$.

We summarize properties of these Floquet operators that are discussed in detail in [31]. For $\alpha \in (0, 1)$ let \mathfrak{A}_α denote the universal rotation C^* -algebra and let \mathfrak{B}_α denote the C^* -algebra of operators on the Hilbert space $L^2(\mathbb{T})$ generated by (multiplication by) $e^{2\pi i t} \in C(\mathbb{T})$ and the rotation operator $(R_\alpha f)(t) = f(t + \alpha)$. \mathfrak{B}_α is isomorphic to \mathfrak{A}_α if α is irrational and for rational $\alpha = p/q$, \mathfrak{B}_α is a homomorphic image of \mathfrak{A}_α and \mathfrak{B}_α is isomorphic to the algebra of matrix valued functions $C(\mathbb{T}, \mathbb{C}^{q \times q})$ acting by pointwise multiplication on the Hilbert space $H = L^2(\mathbb{T}, \mathbb{C}^q)$. If an operator F corresponds to $M \in C(\mathbb{T}, SU(q))$ then

$$\text{spec}(F) = \bigcup_{t \in \mathbb{T}} \text{roots}(zI_q - M(t)), \quad (10)$$

where I_q denotes the $q \times q$ identity matrix. Therefore $\text{spec}(F)$ consists of the union of at most q -disjoint intervals.

1. For rational $\alpha = p/q$, the operators 1, 2, 3 and 5 above belong to \mathfrak{B}_α and their mother operators belong to \mathfrak{A}_α , and the operator 4 belongs to $\mathfrak{B}_{\alpha/2}$ and its mother operators belong to $\mathfrak{A}_{\alpha/2}$.
2. For rational $\alpha = p/q$, the spectrum of $H(\alpha, \lambda, \theta)$ and the spectrum of its mother operator equals the union of q disjoint intervals if q is odd and of $q - 1$ disjoint intervals if q is even, ([9], Theorems 2 and 3), ([12], Theorem 4.7).
3. The KH operator differs from the UH operator by $O(\kappa^2)$ so the spectral mapping theorem implies that for fixed rational α the spectrum of the kicked Harper operator has the same number of bands (disjoint intervals) as described in 2.
4. For irrational α the spectrum of H operator is a Cantor set. This fact was conjectured by Abzel in 1964 [6] and proved by Avila and Jitomirskaya in 2003 [5].
5. ORDKR operators were discovered in 2007 by Jiangbin Gong and Jiao Wang [21] and further discussed by them and Anders Mouritzen in [39]. They noted that their computed spectrums for rational $\alpha = p/q$ were very close to the spectrums of KH operators with the same parameter values, and that for fixed $\lambda = 1$ and κ the Hausdorff distance between their spectrums converged to 0 as q increased. They also noted that ORDKR systems could be realized using Bose-Einstein condensates, which are described in Appendix B, whereas the KH systems can not be experimentally realized because they require magnetic field strengths five orders of magnitude stronger than the most powerful magnetic fields on Earth (the ones used in MRI devices) and only exist in neutron stars. See ([31], Appendix A: Physical Considerations and Experimental Realizations) for a detailed discussion of these practical considerations.
6. In [31] these relationships were derived using properties of rotation C^* -algebras. It was explained that KH operators and ORDKR operators are proper homomorphic images of their mother operators if α is rational and that they are isomorphic to their mother operators if α is irrational. Furthermore, their mother operators are unitarily equivalent (under an automorphism in the Brenken-Watatani automorphic representation of the modular group $SL(2, Z)$ acting on \mathfrak{A}_α) and therefore their spectrums are equal. For $\alpha = p/q$ it was proved that the Hausdorff distance between their spectrums approaches 0 as q increases.
7. In [31] numerical computation of the spectrum of the ORDKR operator for $\alpha = 2584/4181$ showed that it had a fractal type structure. This supports our conjecture that the spectrum is a Cantor set if α is irrational. We briefly discussed approaches to prove our conjecture.
8. In [40] Cantor type spectrum are observed based on numerical computations for a class of operators different from the five families mentioned above.

Since for sufficiently small κ the multiplicity of roots for polynomial P_κ for the ORDKH operator is ≤ 2 and since the roots of P_κ have modulus 1, Corollary 1 and Theorem 1 imply that P_κ is CR. Therefore if P_κ was LCR for all values of κ , since the braid group elements are homotopy invariants, Conjecture 1 would hold for all P_κ . Unfortunately the LCR property is not invariant under ordinary homotopy. Combining Theorem 4 with modern analytic geometry tools, such as Lojaciwicz's Structure Theorem for Varieties [32], ([27], Theorem 5.2.3) and étale homotopy, may provide invariants for the κ parameterized homotopies to help prove Conjecture 1.

6 Brief History of Bose-Einstein Condensates

Bose-Einstein condensates (BEC) arise when a dilute gas of photons or bosonic atoms are cooled to near absolute zero. Under these conditions a large fraction of particles occupy the lowest energy state and the gas exhibits *weird* quantum behavior that was first predicted for photons by Satyendra Nath Bose [11] in 1924 and extended to bosonic matter by Albert Einstein [18, 19]. In 1995 Eric Cornell and Carl Weiman [14] produced the first BEC, consisting of a gas of rubidium atoms cooled to $1.7 \times 10^{-7} K$, at the University of

Colorado at Bolder NIST-JILA lab for which they shared the 2001 Nobel Prize in Physics with Wolfgang Ketterle at MIT. Later in 1995 BCE were used to experimentally realize quantum kicked rotators [34]. In 2010 Jan Klaers, Julian Schmitt, Frank Vewinger and Martin Weitz [28] produced a photon BEC.

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